The implicit function theorem

We will give a proof of the implicit function theorem based on induction on the number of equations. Let $F_1 = F_1(x_1, \ldots, x_m), \ldots, F_n = F_n(x_1, \ldots, x_m)$ be C^1 functions defined in a common domain $\Omega \subset \mathbb{R}^m$, with $n < m$. We consider the system

$$\begin{aligned} F_1(x_1, \ldots, x_m) &= 0 \\ &\vdots \\ F_n(x_1, \ldots, x_m) &= 0 \end{aligned} \tag{1}$$

and a point $\mathbf{x}_0 = (x_1^0, \ldots, x_m^0) \in \Omega$ for which the set equations hold. The theorem asserts that if

$$\frac{\partial (F_1, \ldots, F_n)}{\partial (x_1, \ldots, x_n)}(\mathbf{x}_0) \neq 0 \tag{2}$$

then there exists a neighborhood $U = V \times W$ of $\mathbf{x}_0$, with V a neighborhood of $(x_1^0, \ldots, x_n^0)$ and W a neighborhood of $(x_{n+1}^0, \ldots, x_m^0)$, such that every solution of (1) in U is of the form

$$\begin{aligned} x_1 &= f_1(x_{n+1}, \ldots, x_m) \\ &\vdots \\ x_n &= f_n(x_{n+1}, \ldots, x_m) \end{aligned} \tag{3}$$

for C^1 functions f_k defined in W with $x_k^0 = f_k(x_{n+1}^0, \ldots, x_m^0)$. The chain rule allows then to obtain the derivatives of the f_k.

We prove first the case $n = 1$, and to simplify notation, we will assume that $m = 2$. Thus, let $F(x, y)$ be a C^1 function defined in Ω, and let $(x_0, y_0) \in \Omega$ be such that $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) \neq 0$. We may assume that $F_y(x_0, y_0) > 0$. We will choose neighborhood of the form $U = I \times J$, where $I = (x_0 - a, x_0 + a)$, $J = (y_0 - b, y_0 + b)$.

(i) $F = 0$ **is a graph over** x

For a, b small we can ensure that $F_y \geq c > 0$ in U, so that $y \to F(x, y)$ is strictly increasing for $y \in J$ and $x \in I$ fixed. Since $F(x_0, y_0) = 0$ then $F(x_0, y_0 - b) < 0$ and $F(x_0, y_0 + b) > 0$. By continuity, and for a small enough, we may assume that

$$F < 0 \text{ in the lower edge of } U \quad , \quad F > 0 \text{ in the upper edge of } U\,.$$

Hence, for each fixed $x \in I$ there exists a unique $y = f(x) \in J$ such that

$$F(x, f(x)) = 0\,.$$

The argument shows that these are the only solution of $F = 0$ in U.

(ii) The function $y = f(x)$ is C^1

Let $x \in I$ and for small h let $k = f(x+h) - f(x)$. The mean value theorem ensures the existence of $\theta_1, \theta_2 \in (0,1)$ such that

$$F(x+h, f(x)) = F(x+h, f(x)) - F(x, f(x)) = F_x(x+\theta_1 h, f(x))h\,,$$

and on the other hand

$$F(x+h, f(x)) = F(x+h, f(x)) - F(x+h, f(x+h)) = -F_y(x+h, f(x)+\theta_2 k)k\,,$$

so that

$$k = -\frac{F_x(x+\theta_1 h, f(x))}{F_y(x+h, f(x)+\theta_2 k)}\,h\,. \tag{4}$$

The denominator F_y stays away from 0 in U, while the numerator F_x is continuous, therefore, $k \to 0$ as $h \to 0$. Thus, $y = f(x)$ is continuous, so that for $h \to 0$

$$\frac{k}{h} = -\frac{F_x(x+\theta_1 h, f(x))}{F_y(x+h, f(x)+\theta_2 k)} \to -\frac{F_x(x, f(x))}{F_y(x, f(x))}\,.$$

This proves that the function f is C^1 (as well as giving for the derivative the same expression that yields implicit differentiation).

(iii) The inductive step

We analyze the system (1) with the condition (2), and consider the last equation $F_n = 0$. By (2), there exists $i \in \{1, \ldots, n\}$ for which $(\partial_i F_n)(\mathbf{x}_0) \neq 0$. Without loss of generality, we may assume that this holds for $i = n$. It follows from the case of one equation that in a neighborhood U of $\mathbf{x}_0$ the solutions of $F_n = 0$ correspond to a C^1 graph

$$x_n = g(x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_m)\,, \tag{5}$$

with g defined in a neighborhood of the point $\dot{\mathbf{x}}_0$ given by $\mathbf{x}_0$ with the n-th component removed. We insert this back in (1) to obtain a system of $n-1$ equations in $m-1$ variables:

$$\begin{array}{rcl} G_1 = F_1(x_1, \ldots, x_{n-1}, g, x_{n+1}, \ldots, x_m) & = & 0 \\ \vdots & & \\ G_{n-1} = F_{n-1}(x_1, \ldots, x_{n-1}, g, x_{n+1}, \ldots, x_m) & = & 0 \end{array}\,.$$

We must show now that

$$\frac{\partial\,(G_1, \ldots, G_{n-1})}{\partial\,(x_1, \ldots, x_{n-1})}(\dot{\mathbf{x}}_0) \neq 0\,.$$

For $1 \le i, k \le n-1$ we have

$$\frac{\partial G_k}{\partial x_i} = \frac{\partial F_k}{\partial x_i} + \frac{\partial F_k}{\partial x_n}\frac{\partial g}{\partial x_i} = \frac{\partial F_k}{\partial x_i} - \alpha_i \frac{\partial F_k}{\partial x_n}\,,$$

where

$$\alpha_i = \frac{\partial_i F_n}{\partial_n F_n}\,.$$

Written in columns

$$C_i = \begin{bmatrix} \partial_i G_1 \\ \vdots \\ \partial_i G_{n-1} \end{bmatrix} = \begin{bmatrix} \partial_i F_1 \\ \vdots \\ \partial_i F_{n-1} \end{bmatrix} - \alpha_i \begin{bmatrix} \partial_n F_1 \\ \vdots \\ \partial_n F_{n-1} \end{bmatrix} = A_i - \alpha_i A_n\,.$$

Therefore

$$\det(C_1, \dots, C_{n-1}) = \det(A_1, \dots, A_{n-1}) - \sum_{i=1}^{n-1} \alpha_i \det(A_1, \dots, A_{i-1}, A_n, A_{i+1}, \dots, A_{n-1})\,.$$

The alternating property of the determinant gives

$$\det(A_1, \dots, A_{i-1}, A_n, A_{i+1}, \dots, A_{n-1}) = (-1)^{n-1-i}\det(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{n-1}, A_n)\,,$$

and using the expression for α_i we see that

$$\det(C_1, \dots, C_{n-1}) = (-1)^n (\partial_n F_n)^{-1} \sum_{i=1}^{n} (-1)^i (\partial_i F_n) \det(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{n-1}, A_n)\,.$$

The last sum corresponds to the determinant in (2) as calculated from the n-th row, and thus, $\det(C_1, \dots, C_{n-1}) \neq 0$. The induction hypothesis yields C^1 functions f_k defined in some neighborhood W of $(x^0_{n+1}, \dots, x^0_m)$ such that

$$\begin{aligned} x_1 &= f_1(x_{n+1}, \dots, x_m) \\ &\vdots \\ x_{n-1} &= f_{n-1}(x_{n+1}, \dots, x_m) \end{aligned} \tag{3}$$

represents the solutions of the system $G_1 = \cdots = G_{n-1} = 0$ in a neighborhood $V' \times W$ of $\dot{\mathbf{x}}_0$. This inserted back into (5) completes the proof.